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On the Paper "Asymptotics for the Moments of Singular Distributions"

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In their 1993 paper, W. Goh and J. Wimp derive interesting asymptotics for the moments

$$c_n(\alpha) \equiv c_n = \int_0^1 t^n \, d\alpha(t), \qquad n = 0, 1, 2, ...,$$

of some singular distributions α (with support $\subset [0, 1]$), which contain oscillatory terms. They suspect, that this is a general feature of singular distributions and that this behavior provides a striking contrast with what happens for absolutely continuous distributions. In the present note, however, we give an example of an absolutely continuous measure with asymptotics of moments containing oscillatory terms, and an example of a singular measure having very regular asymptotic behavior of its moments.

Finally, we give a short proof of the fact that the drop-off rate of the moments is exactly the local measure dimension about 1 (if it exists).

1. INTRODUCTION

Suppose we are given a measure μ on the real line. It is uniquely determined by the distribution

$$\alpha(x) = \mu((-\infty, x)) \qquad \text{for} \quad x \in R^1.$$

This function is known to be non-decreasing and left continuous. The support of α (or μ) is defined by

$$\operatorname{Supp} d\alpha = \{x \in R^1 : \alpha(x+\delta) - \alpha(x-\delta) > 0 \ \forall \delta > 0\}.$$

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In this paper, we always will have $\operatorname{Supp} d\alpha \subset [0, 1]$ (and $1 \in \operatorname{Supp} d\alpha$). Our aim is to investigate the asymptotic behavior of the moments

$$c_n(\alpha) \equiv c_n = \int_0^1 t^n \, d\alpha(t), \qquad n = 0, 1, 2, ..., \tag{1}$$

for some singular distributions α (i.e., $\alpha'(x) = 0$ a.e. with respect to Lebesgue measure). This behavior should be determined completely by the local behavior of the distribution α around $x = 1$. In [3] it is conjectured that the drop-off rate of the moments is connected with the local measure dimension of μ at this point. The local measure dimension of μ at x is defined by

$$\lim_{\rho \to 0} \frac{\ln \mu(B_\rho(x))}{\ln \rho},$$

where $B_\rho(x)$ is the ball centered at x with radius ρ. In the present paper, we are able to prove this hypothesis.

Obviously, the local properties of the measure at a single point do not depend on singularity of the measure. Indeed, we are able to give

- an example of an absolutely continuous distribution with moments

$$c_n \sim \frac{1}{n} \left\{ 2\left(2 - \frac{\ln 3}{\ln 2}\right) - \sum_{m=-\infty}^{\infty}{}' \frac{1}{m\pi i} \left(3^{2m\pi i/\ln 2} - 1\right) \Gamma\left(1 - \frac{2m\pi i}{\ln 2}\right) n^{2m\pi i/\ln 2} \right\},$$

- an example of a singular distribution with moments

$$c_n = \frac{1}{n+1} + O(n^{1/4} e^{-2\sqrt{n}}).$$

In [3], the asymptotics of orthogonal polynomials with respect to singular measures (especially of the coefficients in their three term recurrence relation) are investigated, too. This rather complicated problem is beyond the scope of the present note. Some of our results, which confirm a conjecture in [3] concerning convergence of arithmetic means of the recurrence coefficients, will be published elsewhere ([2, Theorem 1]).

2. Some Helpful Lemmas

As in [3], we derive the asymptotics for the moments in our first example from the asymptotics for their exponential generating function. However, we do not use Mellin transform. The following two lemmas would have simplified the proofs in [3] considerably.

LEMMA 1. *Let*

$$\tilde{c}_n(\alpha) \equiv \tilde{c}_n = \int_0^1 e^{n(t-1)}\, d\alpha(t), \qquad n \in R^1, \tag{2}$$

then

$$0 \leqslant \tilde{c}_n - c_n \leqslant \frac{4e^{-2}}{\delta^2 n} \tilde{c}_{n(1-\delta)} \qquad \textit{for any} \quad \delta \in (0, 1]. \tag{3}$$

Proof. First, we have the elementary inequality

$$e^x \geqslant 1 + x \qquad \text{for all} \quad x \in R^1.$$

From this we get with $x = t - 1$

$$e^{t-1} \geqslant t \qquad \text{for all} \quad t \in R^1$$

and

$$te^{-t} \leqslant e^{-1} \qquad \text{for all} \quad t \in R^1.$$

With $x = 1 - t$ we have

$$e^{1-t} \geqslant 2 - t$$

and

$$1 - te^{1-t} \leqslant 1 - t(2-t) = (1-t)^2 \qquad \text{for} \quad t \geqslant 0.$$

Consequently, for any $\delta \in (0, 1]$

$$\begin{aligned} e^{n(t-1)} - t^n &= e^{n(t-1)}[1 - (te^{1-t})^n] \leqslant e^{n(t-1)}\, n(1 - te^{1-t}) \\ &\leqslant e^{n(t-1)} n(1-t)^2 = \frac{4}{\delta^2 n} \left[\frac{n\delta(1-t)}{2} e^{-n\delta(1-t)/2} \right]^2 e^{n(1-\delta)(t-1)}. \end{aligned}$$

But

$$\frac{n\delta(1-t)}{2} e^{-n\delta(1-t)/2} \leqslant e^{-1},$$

and since

$$e^{n(t-1)} \geqslant t^n,$$

we obtain

$$0 \leqslant e^{n(t-1)} - t^n \leqslant \frac{4e^{-2}}{\delta^2 n} e^{n(1-\delta)(t-1)}.$$

From this estimate (3) follows immediately by integration. ∎

Obviously, the sequence $(\tilde{c}_n)$ is closely related to the exponential generating function of (c_n): With

$$f(x, d\alpha) \equiv f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = \int_0^1 e^{tx}\, d\alpha(t)$$

we have

$$\tilde{c}_n = f(n)\, e^{-n}.$$

For the simple type of self-similar distributions studied here, we obtain explicit expressions for $f(x)$ or $\ln f(x)$ as (very special) infinite sums. Their asymptotic behavior is given by

LEMMA 2. *Let* $G \in C^1[0, 1]$, $a > 1$. *Then*

$$\begin{aligned} S(x) &= \sum_{k=1}^{\infty} [G(e^{-a^{-k}x}) - G(1)] \\ &= C_0 + \sum_{m=-\infty}^{\infty}{}' C_m x^{2m\pi i/\ln a} - \frac{G(1)-G(0)}{\ln a} \ln x \\ &\quad + O(e^{-x}) \qquad \textit{for} \quad x \to \infty, \end{aligned}$$

where

$$C_0 = \frac{1}{\ln a}\left[\int_0^1 \frac{G(e^{-x}) - G(1)}{x}\, dx + \int_1^{\infty} \frac{G(e^{-x}) - G(0)}{x}\, dx\right] + \frac{G(1)-G(0)}{2} \tag{4}$$

$$= \frac{1}{\ln a}\int_0^{\infty} \ln x\, e^{-x} G'(e^{-x})\, dx + \frac{G(1)-G(0)}{2} \tag{5}$$

and

$$C_m = -\frac{1}{2m\pi i}\int_0^{\infty} e^{-x} G'(e^{-x})\, x^{-2m\pi i/\ln a}\, dx \qquad \textit{for} \quad m \neq 0. \tag{6}$$

Proof. The trick here is to consider the auxiliary function

$$\tilde{S}(x) = \sum_{k=1}^{\infty} [G(e^{-a^{-k}x}) - G(1)] + \sum_{k=0}^{\infty} [G(e^{-a^{k}x}) - G(0)] + \frac{G(1) - G(0)}{\ln a} \ln x.$$

By our assumptions, we have $|G'(x)| \leqslant C$, and from this immediately follows that the series

$$\sum_{k=1}^{\infty} a^{-k} e^{-a^{-k}x} G'(e^{-a^{-k}x})$$

converges uniformly for $|x| < x_1 < \infty$, and the series

$$\sum_{k=0}^{\infty} a^{k} e^{-a^{k}x} G'(e^{-a^{k}x})$$

converges uniformly for $x > x_0 > 0$. Thus, $\tilde{S}(x)$ is a C^1-function in $[x_0, x_1]$ (for instance, in $[1, a]$), and we see easily

$$\tilde{S}(ax) = \tilde{S}(x).$$

Hence, $\tilde{S}(x)$ is a periodic C^1-function of $u = \ln x$ with period $\ln a$, and therefore it's the sum of the Fourier series (in complex form)

$$\tilde{S}(x) = \sum_{m=-\infty}^{\infty} C_m e^{(2m\pi i/\ln a)u} = \sum_{m=-\infty}^{\infty} C_m x^{2m\pi i/\ln a}$$

with coefficients

$$C_m = \frac{1}{\ln a} \int_0^{\ln a} \tilde{S}(e^u) e^{-(2m\pi i/\ln a)u} du = \frac{1}{\ln a} \int_1^a \tilde{S}(x) x^{-(2m\pi i/\ln a) - 1} dx.$$

In virtue of uniform convergence, we can evaluate the coefficients by termwise integration. If $m \neq 0$, we can omit the constant terms $G(0)$ and $G(1)$, since

$$\int_1^a x^{-(2m\pi i/\ln a) - 1} dx = 0$$

(remember $a^{-2m\pi i/\ln a} = 1!$). Thus, we have

$$C_m = \frac{1}{\ln a} \sum_{k=-\infty}^{\infty} \int_1^a G(e^{-a^k x})\, x^{-(2m\pi i/\ln a)-1}\, dx$$
$$+ \frac{G(1)-G(0)}{\ln^2 a} \int_1^a \ln x \; x^{-(2m\pi i/\ln a)-1}\, dx. \tag{7}$$

But the substitution $x \to a^{-k}x$ and partial integration give

$$\int_1^a G(e^{-a^k x})\, x^{-(2m\pi i/\ln a)-1}\, dx$$
$$= \int_{a^k}^{a^{k+1}} G(e^{-x})\, x^{-(2m\pi i/\ln a)-1}\, dx$$
$$= \left[-\frac{\ln a}{2m\pi i} G(e^{-x})\, x^{-2m\pi i/\ln a} \right]_{a^k}^{a^{k+1}}$$
$$- \frac{\ln a}{2m\pi i} \int_{a^k}^{a^{k+1}} e^{-x} G'(e^{-x})\, x^{-2m\pi i/\ln a}\, dx$$
$$= -\frac{\ln a}{2m\pi i} [G(e^{-a^{k+1}}) - G(e^{-a^k})]$$
$$- \frac{\ln a}{2m\pi i} \int_{a^k}^{a^{k+1}} e^{-x} G'(e^{-x})\, x^{-2m\pi i/\ln a}\, dx,$$

and the sum over this from $k = -\infty$ to $k = \infty$ telescopes into

$$-\frac{\ln a}{2m\pi i} [G(0) - G(1)] - \frac{\ln a}{2m\pi i} \int_0^\infty e^{-x} G'(e^{-x})\, x^{-2m\pi i/\ln a}\, dx. \tag{8}$$

By partial integration again, we find

$$\int_1^a \ln x \; x^{-(2m\pi i/\ln a)-1}\, dx$$
$$= -\frac{\ln a}{2m\pi i} [\ln x \; x^{-2m\pi i/\ln a}]_1^a + \frac{\ln a}{2m\pi i} \int_1^a x^{-(2m\pi i/\ln a)-1}\, dx = -\frac{\ln^2 a}{2m\pi i},$$

together with (8) from (7) follows (6). The case $m = 0$ is similiar: First we have

$$\int_1^a \frac{G(e^{-a^{-k}x}) - G(1)}{x}\, dx = \int_{a^{-k}}^{a^{-k+1}} \frac{G(e^{-x}) - G(1)}{x}\, dx$$

and consequently

$$\sum_{k=1}^{\infty} \int_1^a \frac{G(e^{-a^{-k}x}) - G(1)}{x}\, dx = \int_0^1 \frac{G(e^{-x}) - G(1)}{x}\, dx.$$

Analogously, the second sum evaluates as

$$\sum_{k=0}^{\infty} \int_1^a \frac{G(e^{-a^k x}) - G(0)}{x}\, dx = \int_1^{\infty} \frac{G(e^{-x}) - G(0)}{x}\, dx.$$

Since

$$\int_1^a \ln x \; x^{-1}\, dx = \tfrac{1}{2} \ln^2 a,$$

we get (4). Integrating by parts, from (4) follows (5). Now the proposition follows immediately from

$$S(x) = \tilde{S}(x) - \sum_{k=0}^{\infty} [G(e^{-a^k x}) - G(0)] - \frac{G(1) - G(0)}{\ln a} \ln x$$

and the obvious estimate

$$\left| \sum_{k=0}^{\infty} [G(e^{-a^k x}) - G(0)] \right| \leqslant \sum_{k=0}^{\infty} C e^{-a^k x} = O(e^{-x}). \quad \blacksquare$$

Our last goal is to connect the asymptotics for the moments of $\alpha(t)$ with its local behavior about $t = 1$. To do this, we need

LEMMA 3. *For $n > 0$, we have*

$$c_n = \int_0^1 t^n \, d\alpha(t) = n \int_0^1 t^{n-1} [\alpha(1) - \alpha(t)]\, dt. \tag{9}$$

Proof. This is just the usual formula of partial integration, which is valid in the case of a Stieltjes integral, too (since the integrand t^n is a monotone and continuous function):

$$\begin{aligned} \int_0^1 t^n \, d\alpha(t) &= -\int_0^1 t^n \, d[\alpha(1) - \alpha(t)] \\ &= [-t^n[\alpha(1) - \alpha(t)]]_0^1 + n \int_0^1 t^{n-1} [\alpha(1) + \alpha(t)]\, dt \\ &= n \int_0^1 t^{n-1} [\alpha(1) - \alpha(t)]\, dt. \quad \blacksquare \end{aligned}$$

3. The Results

Now we are able to formulate our results. Thus, we show that asymptotics for moments of absolutely continuous distributions may have oscillating terms. We define $\alpha(t)$ by

$$\alpha(t)=\begin{cases}2t & \text{for} \quad t\in[0,\frac{1}{4}],\\ \frac{1}{2} & \text{for} \quad t\in[\frac{1}{4},\frac{1}{2}],\\ \frac{1}{2}+\frac{1}{2}\alpha(2t-1) & \text{for} \quad t\in[\frac{1}{2},1].\end{cases} \tag{10}$$

It's easy to see that $\alpha'(t)$ exists almost everywhere, in fact we have $\alpha'(t)=2$ for $t\in(1-2^{-k},1-3\cdot 2^{-k-2})$, $k=0,1,2,\ldots$, and $\alpha'(t)=0$ elsewhere. From this we obtain the "explicit" formula

$$c_n=\int_0^1 t^n\,d\alpha(t)=\frac{2}{n+1}\sum_{k=0}^{\infty}\left[\left(1-\frac{3}{2^{k+2}}\right)^{n+1}-\left(1-\frac{1}{2^k}\right)^{n+1}\right].$$

This looks rather innocent, but there are oscillatory terms!

Theorem 1. *Let $\alpha(t)$ be defined by (10). Then we have*

$$c_n\sim\frac{1}{n}\left\{2\left(2-\frac{\ln 3}{\ln 2}\right)-\sum_{m=-\infty}^{\infty}{}'\,\frac{1}{m\pi i}\left(3^{2m\pi i/\ln 2}-1\right)\Gamma\left(1-\frac{2m\pi i}{\ln 2}\right)n^{2m\pi i/\ln 2}\right\}.$$

Proof. From the definition (10) we derive the functional equation

$$\begin{aligned}f(x)&=\int_0^1 e^{tx}\,d\alpha(t)=2\int_0^{1/4}e^{tx}\,dt+\frac{1}{2}\int_{1/2}^1 e^{tx}\,d\alpha(2t-1)\\&=2\,\frac{e^{x/4}-1}{x}+\frac{1}{2}\int_0^1 e^{((t+1)/2)x}\,d\alpha(t)\\&=2\,\frac{e^{x/4}-1}{x}+\frac{1}{2}e^{x/2}f\left(\frac{x}{2}\right).\end{aligned}$$

Multiplying this by xe^{-x}, we get

$$xe^{-x}f(x)=2(e^{-3x/4}-e^{-x})+\frac{x}{2}e^{-x/2}f\left(\frac{x}{2}\right)$$

and iterating this equation

$$xe^{-x}f(x)=2\sum_{k=1}^{\infty}\left(e^{-3\cdot 2^{-k-1}x}-e^{-2^{-k+1}x}\right).$$

Now we apply to the sum our Lemma 2 with $a=2$ and $G(x)=2(x^{3/2}-x^2)$. In this simple case we have $G(0)=G(1)=0$ and $e^{-x}G'(e^{-x})=2(\frac{3}{2}e^{-(3/2)x}-2e^{-2x})$, and by (6) (after some algebra)

$$C_m=-\frac{1}{m\pi i}(3^{2m\pi i/\ln 2}-1)\,\Gamma\left(1-\frac{2m\pi i}{\ln 2}\right) \qquad \text{for} \quad m\neq 0.$$

The coefficient C_0 can be calculated by (4), and we obtain (since the Frullani integral

$$\int_0^{\infty}\frac{e^{-(3/2)x}-e^{-2x}}{x}\,dx$$

is known to be $\ln\frac{4}{3}=2\ln 2-\ln 3$)

$$C_0=2\left(2-\frac{\ln 3}{\ln 2}\right).$$

This means

$$\begin{aligned}\tilde{c}_n&=f(n)\,e^{-n}\\ &=\frac{1}{n}\left\{2\left(2-\frac{\ln 3}{\ln 2}\right)-\sum_{m=-\infty}^{\infty}{}'\,\frac{1}{m\pi i}(3^{2m\pi i/\ln 2}-1)\,\Gamma\left(1-\frac{2m\pi i}{\ln 2}\right)n^{2m\pi i/\ln 2}\right\}\\ &\quad+O\left(\frac{1}{n}e^{-n}\right).\end{aligned}$$

But (3) shows $\tilde{c}_n-c_n=O(1/n^2)$, and this proves our theorem. ∎

After we have shown that there are absolutely continuous distributions with strange asymptotics for the moments, we add an example of a singular distribution with very regular asymptotics for its moments. The idea is to use (9) to enclose a singular distribution between two very near and regular distributions with known moments. Indeed, if we have three distributions $\alpha_0(t)$, $\alpha_1(t)$ and $\alpha(t)$ with $\alpha_0(t)\leqslant\alpha(t)\leqslant\alpha_1(t)$ and $\alpha_0(1)=\alpha(1)=\alpha_1(1)$, from (9) immediately follows $c_n(\alpha_1)\leqslant c_n(\alpha)\leqslant c_n(\alpha_0)$. This simple fact together with Fig. 1 shows the construction clearly. The details are given by

THEOREM 2. *Let (t_n) be defined by $t_0=0$, $t_1=\frac{1}{2}e^{-1}$ and*

$$t_{n+2}=t_n+e^{1/(t_n-1)} \qquad \textit{for} \quad n\geqslant 0,$$

and $\beta(t)$ any singular distribution with $\beta(1)=1$. We define $\alpha(t)$ as

$$\alpha(t)=t_{n+1}+(t_{n+2}-t_{n+1})\,\beta\left(\frac{t-t_n}{t_{n+1}-t_n}\right) \qquad \textit{for} \quad t\in[t_n,t_{n+1}],\ n\geqslant 0.$$

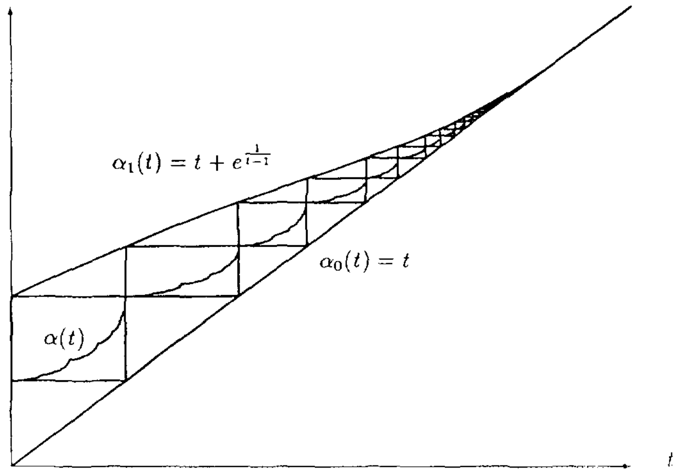

FIGURE 1

Then

$$c_n(\alpha) = \frac{1}{n+1} + O(n^{1/4} e^{-2\sqrt{n}}).$$

Proof. From $0 = t_0 \leqslant t_1 = \frac{1}{2} e^{-1} \leqslant t_2 = e^{-1}$ we have by induction $t_n \leqslant t_{n+1}$ for all $n \geqslant 0$. Let $\alpha_0(t) = t$ and $\alpha_1(t) = t + e^{1/(t-1)}$. For $t \in [t_n, t_{n+1}]$ we have by definition

$$t_{n+1} \leqslant \alpha(t) \leqslant t_{n+2},$$

and by monotony

$$\alpha_0(t) \leqslant \alpha_0(t_{n+1}) = t_{n+1} \leqslant \alpha(t) \leqslant t_{n+2} = \alpha_1(t_n) \leqslant \alpha_1(t).$$

Consequently, $\lim_{n \to \infty} t_n = 1$, and $\alpha(t)$ is well-defined for any $t \in [0, 1)$. Of course, we set

$$\alpha(1) = \lim_{n \to \infty} \alpha(t_n) = \lim_{n \to \infty} t_{n+2} = 1.$$

Thus, we have $\alpha_0(t) \leqslant \alpha(t) \leqslant \alpha_1(t)$ and $\alpha_0(1) = \alpha(1) = \alpha_1(1)$, and from this follows

$$c_n(\alpha_1) \leqslant c_n(\alpha) \leqslant c_n(\alpha_0)$$

and

$$0 \leqslant c_n(\alpha_0) - c_n(\alpha) \leqslant c_n(\alpha_0) - c_n(\alpha_1) = \int_0^1 t^n \frac{1}{(1-t)^2} e^{1/(t-1)}\, dt.$$

But we can estimate

$$\int_0^1 t^n \frac{1}{(1-t)^2} e^{1/(t-1)}\, dt \leqslant \int_{-\infty}^1 \frac{1}{(1-t)^2} e^{n(t-1)+1/(t-1)}\, dt,$$

and the last integral with the substitution $t = 1 - (1/\sqrt{n})\, e^{-u}$ transforms into

$$\begin{aligned}
&\sqrt{n} \int_{-\infty}^{\infty} e^u e^{-2\sqrt{n}\cosh u}\, du \\
&= \sqrt{n} \int_{-\infty}^{\infty} (\cosh u + \sinh u)\, e^{-2\sqrt{n}\cosh u}\, du \\
&= \sqrt{n} \int_{-\infty}^{\infty} \cosh u e^{-2\sqrt{n}\cosh u}\, du \\
&= 2\sqrt{n} \int_0^{\infty} \cosh u e^{-2\sqrt{n}\cosh u}\, du = 2\sqrt{n} K_1(2\sqrt{n})
\end{aligned}$$

(see [1, 9.6.24]). The asymptotics of the modified Bessel functions K_ν are well-known: From [1, 9.7.2] with $\nu = 1$ and $z = 2\sqrt{n}$ we get

$$2\sqrt{n} K_1(2\sqrt{n}) \sim \sqrt{\pi} n^{1/4} e^{-2\sqrt{n}},$$

and this proves the result. ∎

Now we connect the drop-off rate of moments with the local measure dimension around 1:

THEOREM 3. *Let μ be the measure defined by the distribution α and $B_\rho(x)$ the ball centered at x with radius ρ. We assume that*

$$\lim_{\rho \to 0} \frac{\ln \mu(B_\rho(1))}{\ln \rho} = \gamma$$

exists. Then

$$\lim_{n \to \infty} \frac{\ln c_n}{\ln n} = -\gamma. \tag{11}$$

Proof. In our situation, $B_\rho(x) = [x-\rho, x+\rho]$ and (recall Supp $d\alpha \subset [0, 1]$!)

$$\mu(B_\rho(1)) = \alpha(1) - \alpha(1-\rho).$$

Let $\varepsilon > 0$. By our assumption,

$$\gamma - \varepsilon \leqslant \frac{\ln(\alpha(1) - \alpha(t))}{\ln(1-t)} \leqslant \gamma + \varepsilon$$

or

$$(1-t)^{\gamma+\varepsilon} \leqslant \alpha(1) - \alpha(t) \leqslant (1-t)^{\gamma-\varepsilon} \tag{12}$$

for $t > 1-\delta$. From (9) follows

$$\begin{aligned} c_n &= \int_0^1 nt^{n-1}[\alpha(1) - \alpha(t)]\, dt \\ &= \int_{1-\delta}^1 nt^{n-1}[\alpha(1) - \alpha(t)]\, dt + O((1-\delta)^n), \end{aligned}$$

and for any $\beta > 0$ we have

$$\begin{aligned} &\int_{1-\delta}^1 nt^{n-1}(1-t)^\beta\, dt \\ &\quad = \int_0^1 nt^{n-1}(1-t)^\beta\, dt + O((1-\delta)^n) \\ &\quad = n\frac{\Gamma(\beta+1)\,\Gamma(n)}{\Gamma(n+\beta+1)} + O((1-\delta)^n) \\ &\quad = \frac{\Gamma(\beta+1)\,\Gamma(n+1)}{\Gamma(n+\beta+1)} + O((1-\delta)^n). \end{aligned}$$

Together with (12) this gives

$$\begin{aligned} &\frac{\Gamma(\gamma+\varepsilon+1)\,\Gamma(n+1)}{\Gamma(n+\gamma+\varepsilon+1)} + O((1-\delta)^n) \\ &\quad \leqslant c_n \leqslant \frac{\Gamma(\gamma-\varepsilon+1)\,\Gamma(n+1)}{\Gamma(n+\gamma-\varepsilon+1)} + O((1-\delta)^n). \end{aligned}$$

Observing

$$\frac{\Gamma(n+1)}{\Gamma(n+\beta+1)} \sim n^{-\beta}$$

as $n \to \infty$ (see [1, 6.1.46]), we immediately obtain

$$-\gamma - \varepsilon \leqslant \liminf_{n \to \infty} \frac{\ln c_n}{\ln n} \leqslant \limsup_{n \to \infty} \frac{\ln c_n}{\ln n} \leqslant -\gamma + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done. ▮

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